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FIRST GAP SIZE OF COEFFICIENTS OF A MODULAR FORM

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Abstract. We find an upper bound on the least positive exponent of a nonzero Fourier coefficient for a modular form with a nonzero constant term for $\Gamma_0(N)$ under certain conditions where N is a prime number p or $2p$.

1. Introduction

Cal Ludwing Siegel [3] found an upper bound on the least positive exponent of a nonzero Fourier coefficient for any level-one modular form with a nonzero constant term. Following Siegel and Brent [1, 2] found upper bounds for the first positive exponent of a nonzero Fourier coefficient occurring in the expansion at infinity of a modular form with a nonzero constant term for $\Gamma_0(2)$. There, the whole Siegel argument carried. Their Fourier coefficients code up representation numbers of quadratic forms.

In this paper, by using results of Siegel and Brent we find an upper bound on the least positive exponent of a nonzero Fourier coefficient for a modular form with a nonzero constant term for $\Gamma_0(N)$ under certain conditions where N is a prime number p or $2p$ (Theroem 2.1, 2.3).

2. Bounds for gaps in Fourier expansions

As usual, for a positive integer N we denote by $\Gamma_0(N)$ the congruence subgroup

$$\Gamma_0(N) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \}$$

By $M_k(\Gamma_0(N))$ we denote the vector space of modular forms for $\Gamma_0(N)$.

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The dimension of $M_k(\Gamma_0(N))$ is denoted by $r(N,k)$. We have the following formulas for positive even number k:

$$r(1,k) = \begin{cases} [\frac{k}{12}]+1, & k \not\equiv 2 \pmod{12} \\ [\frac{k}{12}], & k \equiv 2 \pmod{12}. \end{cases}$$

For any positive even k,

$$r(2,k) = [\frac{k}{4}]+1.$$

Let $q = e^{2\pi iz}$ and $m_k^{(N)}$ be the maximum of $m(g)$, where $g \in M_k(\Gamma_0(N))$ runs through $g = 1 + \sum_{n=m(g)}^{\infty} a_g(n)q^n$ with $a_g(m(g)) \neq 0$. Then by the work [3, Satz 2] of Siegel we have $m_k^{(1)} = r(1,k)$.

Now we are ready to see the main theorem as follows.

Theorem 2.1. *Let p be any prime number. Then for each k satisfying one of the following:*

(1) $k \equiv 0 \pmod{12}$ *and* $12p > k$, *(2)* $k \equiv 4 \pmod{12}$ *and* $8p > k$,
(3) $k \equiv 6 \pmod{12}$ *and* $6p > k$, *(4)* $k \equiv 8 \pmod{12}$ *and* $4p > k$,
(5) $k \equiv 10 \pmod{12}$ *and* $2p > k$,

we have that

$$m_k^{(p)} = p \cdot r(1,k)$$

Proof. Let

$$\begin{aligned} f_k^{(1)} &:= 1 + a_1 q^{m_k^{(1)}} + \cdots \in M_k(\Gamma_0(1)) \text{ with } a_1 \neq 0, \\ f_k^{(p)} &:= 1 + a_p q^{m_k^{(p)}} + \cdots \in M_k(\Gamma_0(p)) \text{ with } a_p \neq 0. \end{aligned}$$

Then

$$f_k^{(1)}(pz) = 1 + a_1 q^{pm_k^{(1)}} + \cdots \in M_k(\Gamma_0(p)).$$

Assume that $f_k^{(1)}(pz) \neq f_k^{(p)}$. As is well known the valence formula for any modular form $f = \sum a_f(n)q^n$ for $\Gamma_0(N)$ implies that there exists $n \leq (k/12)[\Gamma(1):\Gamma_0(N)]$ such that $a_f(n) \neq 0$ if $f \neq 0$. From this fact, we obtain that

$$pm_k^{(1)} \leq \frac{k}{12}[\Gamma(1):\Gamma_0(p)] = \frac{k}{12}(1+p).$$

Suppose that $k \equiv 0 \pmod{12}$ and $12p > k$. Then we have that

$$pm_k^{(1)} \leq \frac{k}{12}(1+p) \text{ if and only if } p(\frac{k}{12}+1) \leq \frac{k}{12}(1+p).$$

which implies that $12p \leq k$. This is a contradiction. Hence $f_k^{(1)}(pz) = f_k^{(p)}$ and $m_k^{(p)} = p \cdot r(1,k)$. For other cases, similarly we can obtain the assertion. $\square$

REMARK 2.2. Since the genus of $\Gamma_0(2)$ is zero, by the result [1, 2] of Barry Bent we have $m_k^{(2)} = r(2, k)$. So our results are consistent with that of Barry Brent.

THEOREM 2.3. *Let p be any odd prime number. If* (1)*: $k \equiv 0 \pmod 4$ and $4p > k$ or* (2)*: $k \equiv 2 \pmod 4$ and $2p > k$, then*

$$m_k^{(2p)} = p \cdot r(2, k)$$

Proof. Let

$$f_k^{(2)}(z) \quad := \quad 1 + a_2 q^{m_k^{(2)}} + \cdots \in M_k(\Gamma_0(2)) \text{ with } a_2 \neq 0$$

and

$$f_k^{(2p)}(z) \quad := \quad 1 + a_{2p} q^{m_k^{(2p)}} + \cdots \in M_k(\Gamma_0(2p)) \text{ with } a_{2p} \neq 0$$

Then $f_k^{(2)}(pz)$ are contained in $M_k(\Gamma_0(2p))$. If $f_k^{(2p)}(z) \neq f_k^{(2)}(pz)$, then

$$p(1 + [\frac{k}{4}]) = p m_k^{(2)} \leq \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(2p)] = \frac{k(p+1)}{4} \tag{2.1}$$

If $k \equiv 0 \pmod 4$, then (2.1) shows that $4p \leq k$. This is contradiction. Hence if $k \equiv 0 \pmod 4$ and $4p > k$, then

$$m_k^{(2p)} = p \cdot r(2, k).$$

Similarly we obtain that if $k \equiv 2$ mod4 and $2p > k$, then

$$m_k^{(2p)} = p \cdot r(2, k)$$

□

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